

Fraïssé sequences – a category-theoretic approach to universal homogeneous structures

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Abstract

We present a category-theoretic approach to universal homogeneous objects, with applications in the theory of Banach spaces and in set-theoretic topology.

Disclaimer: This is only a draft, full of gaps and inaccuracies, put to the Math arXiv for the sake of reference. More complete versions are coming soon.

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1 Introduction

[...]

2 Definitions and notation

Categories will usually be denoted by letters \mathfrak{K} , \mathfrak{L} , \mathfrak{M} , etc. Let \mathfrak{K} be a category. We shall write “ $a \in \mathfrak{K}$ ” for “ a is an object of \mathfrak{K} ”. Given $a, b \in \mathfrak{K}$, we shall denote by $\mathfrak{K}(a, b)$ the set of all \mathfrak{K} -morphisms from a to b . The composition of two compatible arrows f and g will be denoted by $g \circ f$. A *subcategory* of \mathfrak{K} is a category \mathfrak{L} such that each object of \mathfrak{L} is an object of \mathfrak{K} and each arrow of \mathfrak{L} is an arrow of \mathfrak{K} (with the same domain and codomain). We write $\mathfrak{L} \subseteq \mathfrak{K}$. Recall that a subcategory \mathfrak{L} of \mathfrak{K} is *full* if $\mathfrak{L}(a, b) = \mathfrak{K}(a, b)$ for every objects $a, b \in \mathfrak{L}$. We say that \mathfrak{L} is *cofinal* in \mathfrak{K} if for every object $x \in \mathfrak{K}$ there exists an object $y \in \mathfrak{L}$ such that $\mathfrak{K}(x, y) \neq \emptyset$. The opposite category to \mathfrak{K} will be denoted by \mathfrak{K}^{op} . That is, the objects of \mathfrak{K}^{op} are the objects of \mathfrak{K} and arrows are reversed, i.e. $\mathfrak{K}^{\text{op}}(a, b) = \mathfrak{K}(b, a)$.

Let \mathfrak{K} be a category. We say that \mathfrak{K} has the *amalgamation property* if for every $a, b, c \in \mathfrak{K}$ and for every morphisms $f \in \mathfrak{K}(a, b)$, $g \in \mathfrak{K}(a, c)$ there exist $d \in \mathfrak{K}$ and morphisms $f' \in \mathfrak{K}(b, d)$ and $g' \in \mathfrak{K}(c, d)$ such that $f' \circ f = g' \circ g$. If, additionally, for every arrows f'', g'' such that $f'' \circ f = g'' \circ g$ there exists a unique arrow h satisfying $h \circ f' = f''$ and $h \circ g' = g''$ then the pair $\langle f', g' \rangle$ is a *pushout* of $\langle f, g \rangle$. Reversing the arrows, we define the *reversed amalgamation* and the *pullback*. We say that \mathfrak{K} has the *joint embedding property* if for every $a, b \in \mathfrak{K}$ there exists $g \in \mathfrak{K}$ such that both sets $\mathfrak{K}(a, g)$, $\mathfrak{K}(b, g)$ are nonempty.

Fix a category \mathfrak{K} and fix an ordinal $\delta > 0$. An *inductive δ -sequence* in \mathfrak{K} is formally a covariant functor from δ (treated as a poset category) into \mathfrak{K} . In other words, it could be described as a pair of the form $\langle \{a_\xi\}_{\xi < \delta}, \{a_\xi^\eta\}_{\xi < \eta < \delta} \rangle$, where δ is an ordinal, $\{a_\xi : \xi < \delta\} \subseteq \mathfrak{K}$ and $a_\xi^\eta \in \mathfrak{K}(a_\xi, a_\eta)$ are such that $a_\eta^\varrho a_\xi^\eta = a_\xi^\varrho$ for every $\xi < \eta < \varrho < \delta$. We shall denote such a sequence shortly by \vec{a} . The ordinal δ is the *length* of \vec{a} .

Let κ be an infinite cardinal. A category \mathfrak{K} is κ -*continuous* if all inductive sequences of length $< \kappa$ have colimits in \mathfrak{K} . Every category is \aleph_0 -continuous, since the colimit of a finite sequence is its last object. More generally, we say that a category \mathfrak{K} is *relatively κ -continuous in \mathfrak{L}* , if $\mathfrak{K} \subseteq \mathfrak{L}$ and every sequence in \mathfrak{K} of length $< \kappa$ has a colimit in \mathfrak{L} . A category \mathfrak{K} is κ -*bounded* if for every inductive sequence \vec{x} in \mathfrak{K} of length $\lambda < \kappa$ there exist $y \in \mathfrak{K}$ and a cocone of arrows $\{y_\alpha\}_{\alpha < \lambda}$ such that $y_\alpha : x_\alpha \rightarrow y$ and $y_\beta \circ x_\alpha^\beta = y_\alpha$ for

every $\alpha < \beta < \lambda$. Obviously, every κ -continuous category is κ -bounded. We shall write “ σ -continuous” and “ σ -bounded” for “ \aleph_1 -continuous” and “ \aleph_1 -bounded” respectively. We shall need the following notion concerning families of arrows. Fix a family of arrows \mathcal{F} in a given category \mathfrak{K} . We shall write $\text{Dom}(\mathcal{F})$ for the set $\{\text{dom}(f) : f \in \mathcal{F}\}$. We say that \mathcal{F} is *dominating* in \mathfrak{K} if the family of objects $\text{Dom}(\mathcal{F})$ is cofinal in \mathfrak{K} and moreover for every $a \in \text{Dom}(\mathcal{F})$ and for every arrow $f : a \rightarrow x$ in \mathfrak{K} there exists an arrow g in \mathfrak{K} such that $g \circ f \in \mathcal{F}$.

2.1 Arrows between sequences

Fix a category \mathfrak{K} and denote by $\text{Seq}_{<\kappa}(\mathfrak{K})$ the class of all sequences in \mathfrak{K} which have length $< \kappa$. We shall write $\text{Seq}_{\leq\kappa}(\mathfrak{K})$ instead of $\text{Seq}_{<\kappa+}(\mathfrak{K})$ and $\sigma\mathfrak{K}$ instead of $\text{Seq}_{\leq\aleph_0}(\mathfrak{K})$.

We would like to turn $\text{Seq}_{<\kappa}(\mathfrak{K})$ into a category in such a way that an arrow from a sequence \vec{a} into a sequence \vec{b} induces an arrow from $\lim \vec{a}$ into $\lim \vec{b}$, whenever \mathfrak{K} is embedded into a category in which sequences \vec{a}, \vec{b} have colimits.

Fix two sequences \vec{a} and \vec{b} in a given category \mathfrak{K} . Let $\lambda = \text{dom}(\vec{a})$, $\varrho = \text{dom}(\vec{b})$. A *transformation* from \vec{a} to \vec{b} is, by definition, a natural transformation from \vec{a} into $\vec{b} \circ \varphi$, where $\varphi : \lambda \rightarrow \varrho$ is an order preserving map (i.e. a covariant functor from λ to ϱ).

In order to define an arrow from \vec{a} to \vec{b} we need to identify some transformations. Fix two natural transformations $F : \vec{a} \rightarrow \vec{b} \circ \varphi$ and $G : \vec{a} \rightarrow \vec{b} \circ \psi$. We shall say that F and G are *equivalent* if the following conditions hold:

- (1) For every $\alpha \leq \beta$ such that $\varphi(\alpha) \leq \psi(\beta)$ we have that $b_{\varphi(\alpha)}^{\psi(\beta)} \circ F(\alpha) = G(\beta) \circ a_\alpha^\beta$.
- (2) For every $\alpha \leq \beta$ such that $\psi(\alpha) \leq \varphi(\beta)$ we have that $b_{\psi(\alpha)}^{\varphi(\beta)} \circ G(\alpha) = F(\beta) \circ a_\alpha^\beta$.

It is rather clear that this defines an equivalence relation. Every equivalence class of this relation will be called an *arrow* (or *morphism*) from \vec{a} to \vec{b} . It is easy to check that this indeed defines a category structure on all sequences in \mathfrak{K} . The identity arrow of \vec{a} is the equivalence class of the identity natural transformation $\text{id}_{\vec{a}} : \vec{a} \rightarrow \vec{a}$.

3 Fraïssé sequences

Below we introduce the key notion of this work.

Let \mathfrak{K} be a category and let κ be a cardinal. A *Fraïssé sequence* of length κ in \mathfrak{K} (briefly: a κ -*Fraïssé sequence*) is an inductive sequence \vec{u} satisfying the following conditions:

- (U) For every $x \in \mathfrak{K}$ there exists $\xi < \kappa$ such that $\mathfrak{K}(x, u_\xi) \neq \emptyset$.
- (A) For every $\xi < \kappa$ and for every arrow $f \in \mathfrak{K}(u_\xi, y)$, where $y \in \mathfrak{K}$, there exist $\eta \geq \xi$ and $g \in \mathfrak{K}(y, u_\eta)$ such that $u_\xi^\eta = g \circ f$.

An inductive sequence satisfying (U) will be called \mathfrak{K} -*cofinal*. More generally, a collection \mathcal{U} of objects of \mathfrak{K} is \mathfrak{K} -*cofinal* if for every $x \in \mathfrak{K}$ there is $u \in \mathcal{U}$ such that $\mathfrak{K}(x, u) \neq \emptyset$. Condition (A) will be called *amalgamation property*.

3.1 Basic properties

Let \vec{v} be a κ -sequence in a category \mathfrak{K} . We say that \vec{v} has the *extension property* if the following holds:

- (E) For every arrows $f: a \rightarrow b$, $g: a \rightarrow v_\alpha$ in \mathfrak{K} , where $\alpha < \kappa$, there exist $\beta \geq \alpha$ and an arrow $h: b \rightarrow v_\beta$ such that $i_\alpha^\beta \circ g = h \circ f$.

Clearly, this condition implies (A).

Proposition 3.1. *Let \vec{u} be a κ -Fraïssé sequence in a category \mathfrak{K} . Then \mathfrak{K} has the joint embedding property. Moreover, the following conditions are equivalent:*

- (a) \vec{u} has the extension property.
- (b) \mathfrak{K} has the amalgamation property.

Proof. The first statement is trivial. Assume (a) and fix arrows $f: z \rightarrow x$, $g: z \rightarrow y$. Using (U), find $h: x \rightarrow u_\alpha$, $\alpha < \kappa$. Using (E), find $\beta \geq \alpha$ and $k: y \rightarrow u_\beta$ such that $k \circ g = u_\alpha^\beta \circ h \circ f$. Thus (b) holds.

Finally, assume (b) and fix arrows $f: a \rightarrow b$ and $g: a \rightarrow u_\alpha$, $\alpha < \kappa$. Using (b), find arrows $f_1: b \rightarrow w$ and $g_1: u_\alpha \rightarrow w$ so that $f_1 \circ f = g_1 \circ g$. Using (A) for the sequence \vec{u} , we find $\beta \geq \alpha$ and $h: w \rightarrow u_\beta$ so that $h \circ g_1 = u_\alpha^\beta$. Thus $(h \circ f_1) \circ f = h \circ g_1 \circ g = u_\alpha^\beta \circ g$, which shows that (a) holds. \square

Proposition 3.2. *Assume \mathfrak{K} is a category with the joint embedding property. Then every sequence in \mathfrak{K} satisfying condition (A) is Fraïssé.*

Proof. Let \vec{u} be a sequence in \mathfrak{K} satisfying (A). Fix $x \in \mathfrak{K}$. Using the joint embedding property, there exist $w \in \mathfrak{K}$ and arrows $f: u_0 \rightarrow w$, $g: x \rightarrow w$. Using (A), we find an arrow $h: w \rightarrow u_\xi$ such that $h \circ f = u_0^\xi$. Thus $\mathfrak{K}(x, u_\xi) \neq \emptyset$, which shows (U). \square

Proposition 3.3. *Let \mathfrak{K} be a category, let \vec{u} be an inductive sequence of length κ in \mathfrak{K} and let $S \subseteq \kappa$ be unbounded in κ .*

- (a) *If \vec{u} is a Fraïssé sequence in \mathfrak{K} then $\vec{u} \upharpoonright S$ is Fraïssé in \mathfrak{K} .*
- (b) *If \mathfrak{K} has amalgamation and $\vec{u} \upharpoonright S$ is a Fraïssé sequence in \mathfrak{K} then so is \vec{u} .*

Proof. Assume \vec{u} is a Fraïssé sequence. Then $\vec{u} \restriction S$ clearly satisfies (U). In order to check (A), fix $f: u_\xi \rightarrow y$ with $\xi \in S$. Then $u_\xi^\eta = g \circ f$ for some arrow g and for some $\eta \geq \xi$. Since S is unbounded in κ , there is $\alpha \in S$ such that $\alpha \geq \eta$. Then $u_\xi^\alpha = u_\eta^\alpha \circ g \circ f$, which shows that $\vec{u} \restriction S$ satisfies (A).

Now assume $\vec{u} \restriction S$ is a Fraïssé sequence. Clearly, \vec{u} satisfies (U). Fix $f: u_\xi \rightarrow y$, $\xi < \kappa$. Find $\alpha \in S$ with $\alpha \geq \xi$. Using the amalgamation property of \mathfrak{K} , find $f': u_\alpha \rightarrow z$ such that the diagram

$$\begin{array}{ccc} u_\alpha & \xrightarrow{f'} & z \\ u_\xi^\alpha \uparrow & & \uparrow g \\ u_\xi & \xrightarrow{f} & y \end{array}$$

commutes for some arrow g in \mathfrak{K} . Now, using (A) for $\vec{u} \restriction S$, we can find $\beta \in S$ such that $\beta \geq \alpha$ and $h \circ f' = u_\alpha^\beta$ holds for some $h: z \rightarrow u_\beta$. This shows that \vec{u} satisfies (A). \square

A Fraïssé sequence can possibly have finite length. In that case, by Proposition 3.3(a), there is also a Fraïssé sequence of length one – it is an object u which is cofinal in \mathfrak{K} and which satisfies the following version of (A): given $f \in \mathfrak{K}(u, x)$, where $x \in \mathfrak{K}$, there exists $g \in \mathfrak{K}(x, u)$ such that $g \circ f = \text{id}_u$. We shall call u a *Fraïssé object* in \mathfrak{K} . Given a Fraïssé object u , the sequence $u \rightarrow u \rightarrow \dots$, where each arrow is identity, is a Fraïssé sequence of length ω . Thus, it follows from Theorem 3.9 below that a possible Fraïssé object is unique, up to isomorphism. Below we give a direct proof of this fact.

Proposition 3.4. *Assume u, v are Fraïssé objects in a category \mathfrak{K} . Then $u \approx v$. If moreover all arrows in \mathfrak{K} are monomorphisms then every arrow $f: u \rightarrow x$ is an isomorphism.*

Proof. Applying (U) for v , we find a morphism $f_0: u \rightarrow v$ which, using (A) for u , has a left inverse $g_0: v \rightarrow u$, i.e. $g_0 \circ f_0 = \text{id}_u$. Now, using (A) for v , we obtain an arrow $f_1: u \rightarrow v$ such that $f_1 \circ g_0 = \text{id}_v$. Observe that

$$f_1 = f_1 \circ \text{id}_u = f_1 \circ (g_0 \circ f_0) = (f_1 \circ g_0) \circ f_0 = \text{id}_v \circ f_0 = f_0.$$

Hence $f_0 \circ g_0 = \text{id}_v$, which shows that f_0 is an isomorphism.

Finally, let $f: u \rightarrow x$ be a morphism in \mathfrak{K} . Again by (A), f has a left inverse $g: x \rightarrow u$. Assuming g is a monomorphism, we deduce that $f \circ g = \text{id}_x$, because $g \circ (f \circ g) = g \circ \text{id}_x$. Thus f is an isomorphism. \square

3.2 The existence

We present below a simple yet useful criterion for the existence of a Fraïssé sequence. In case of sequences of length $\leq \aleph_1$, this criterion becomes a characterization.

Theorem 3.5 (Existence). *Let κ be an infinite regular cardinal and let \mathfrak{K} be a κ -bounded category which has the amalgamation property and the joint embedding property. Assume further that $\mathcal{F} \subseteq \text{Arr}(\mathfrak{K})$ is dominating in \mathfrak{K} and $|\mathcal{F}| \leq \kappa$. Then there exists a Fraïssé sequence \vec{u} of length κ in \mathfrak{K} .*

Proof. Let $\text{Dom}(\mathcal{F}) = \{a_\alpha\}_{\alpha < \kappa}$ and enumerate \mathcal{F} as $\{f_\alpha\}_{\alpha < \kappa}$ so that for each $f \in \mathcal{F}$ the set $\{\alpha : f = f_\alpha\}$ has cardinality κ . We shall construct inductively the sequence \vec{u} , so that the following conditions are satisfied:

- (i) $u_\eta^\alpha \circ u_\xi^\eta = u_\xi^\alpha$ for every $\xi < \eta < \alpha$.
- (ii) $u_\alpha \in \text{Dom}(\mathcal{F})$ and $\mathfrak{K}(a_\alpha, u_\alpha) \neq \emptyset$.
- (iii) Given $\xi < \alpha$, if $\text{dom}(f_\alpha) = u_\xi$ then there exists an arrow h in \mathfrak{K} such that $h \circ f_\alpha = u_\xi^\alpha$.

We start with $u_0 = a_0$. Assume that $\beta < \kappa$ is such that u_ξ and u_ξ^η have been constructed for all $\xi < \eta < \beta$. Using the fact that \mathfrak{K} is κ -bounded, find $v \in \mathfrak{K}$ and $j_\alpha : u_\alpha \rightarrow v$ such that $j_\xi = j_\eta \circ u_\xi^\eta$ holds for every $\xi < \eta < \beta$. Using the joint embedding property, we may ensure that $\mathfrak{K}(a_\beta, v) \neq \emptyset$. Now, if $f_\beta : u_\alpha \rightarrow y$ and $\alpha < \beta$ then using amalgamation we may find arrows $h : v \rightarrow w$ and $g : y \rightarrow w$ so that $g \circ f_\beta = h \circ j_\alpha$ holds. Using (D1), we may further assume that $w \in \text{Dom}(\mathcal{F})$. Finally, set $u_\beta := w$ and $u_\xi^\beta := h \circ j_\xi$ for $\xi < \beta$. It is clear that conditions (i) – (iii) hold.

It follows that the construction can be carried out. It remains to check that $\vec{u} : \kappa \rightarrow \mathfrak{K}$ is a Fraïssé sequence. Condition (i) says that \vec{u} is indeed an inductive sequence. Conditions (D1) and (ii) imply (U). In order to justify (A), fix $\xi < \kappa$ and $f \in \mathfrak{K}(u_\xi, x)$, where $x \in \mathfrak{K}$. We need to find $\alpha > \xi$ and an arrow g so that $g \circ f = u_\xi^\alpha$. Since $u_\xi \in \text{Dom}(\mathcal{F})$, using (D2), we can find $g \in \mathcal{F}$ such that $g = k \circ f$ for some arrow k . Now find $\alpha > \xi$ such that $f_\alpha = g$. By (iii), $h \circ g = u_\xi^{\alpha+1}$ for some arrow h . Hence $(h \circ k) \circ f = u_\xi^{\alpha+1}$, which completes the proof. \square

3.3 Cofinality

Below we discuss the crucial property of a Fraïssé sequence: cofinality in the category of sequences.

Theorem 3.6 (Countable Cofinality). *Assume \vec{u} is a Fraïssé sequence in a category with amalgamation \mathfrak{K} . Then for every countable inductive sequence \vec{x} in \mathfrak{K} there exists a morphism of sequences $F : \vec{x} \rightarrow \vec{u}$.*

Proof. We use the extension property (property (E)) of the sequence \vec{u} , which is equivalent to the amalgamation property of \mathfrak{K} (Proposition 3.1). Let \vec{x} be an ω -sequence in \mathfrak{K} . Using (U), find an arrow $f_0: x_0 \rightarrow u_{\alpha_0}$. Now assume that arrows f_0, \dots, f_{n-1} have been defined so that $f_m: x_m \rightarrow u_{\alpha_m}$ and the diagram

$$\begin{array}{ccc} x_\ell & \xrightarrow{f_\ell} & u_{\alpha_\ell} \\ x_k^\ell \uparrow & & \uparrow u_{\alpha_k}^{\alpha_\ell} \\ x_k & \xrightarrow{f_k} & u_{\alpha_k} \end{array}$$

commutes for every $k < \ell < n$ (in particular $\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_{n-1}$). Using (E), find $\alpha_n \geq \alpha_{n-1}$ and an arrow $f: x_n \rightarrow u_{\alpha_n}$ so that $f \circ x_{n-1}^n = u_{\alpha_{n-1}}^{\alpha_n} \circ f_{n-1}$ and define $f_n := f$. Given $m < n - 1$, by the induction hypothesis, we get

$$f_n \circ x_m^n = f_n \circ x_{n-1}^n \circ x_m^{n-1} = u_{\alpha_{n-1}}^{\alpha_n} \circ f_{n-1} \circ x_m^{n-1} = u_{\alpha_{n-1}}^{\alpha_n} \circ u_{\alpha_m}^{\alpha_{n-1}} \circ f_m = u_{\alpha_m}^{\alpha_n} \circ f_m.$$

Finally, setting $F = \{f_n\}_{n \in \omega}$, we obtain the required morphism $F: \vec{x} \rightarrow \vec{u}$. \square

The above proof can be easily extended to uncountable sequences, assuming continuity:

Theorem 3.7. *Let \mathfrak{K} be a category with the amalgamation property and let \vec{u} be a Fraïssé sequence of a regular length κ in \mathfrak{K} . Then for every continuous sequence $\vec{x} \in \text{Seq}_{\leq \kappa}(\mathfrak{K})$ there exists an arrow of sequences $F: \vec{x} \rightarrow \vec{u}$.*

Proof. We repeat the construction from the proof of Theorem 3.6. In the case of a limit ordinal δ , we let α_δ to be the supremum of $\{\alpha_\xi: \xi < \delta\}$ and we define f_δ to be the unique arrow satisfying $f_\delta \circ x_\alpha^\delta = f_\alpha$ for every $\alpha < \delta$. This is possible, because x_δ together with the cocone of arrows $\{x_\xi^\delta\}_{\xi < \delta}$ is, by assumption, the colimit of $\vec{x} \upharpoonright \delta$. Thus, the construction from the proof of Theorem 3.6 can be carried out, obtaining the desired arrow $F: \vec{x} \rightarrow \vec{u}$. \square

We shall see later that an uncountable Fraïssé sequence may not be cofinal for ω_1 -sequences. From Theorem 3.6 we immediately get the following characterization of the existence of a Fraïssé sequence of length ω_1 .

Corollary 3.8. *Let \mathfrak{K} be a category with the amalgamation property. There exists a Fraïssé sequence of length ω_1 in \mathfrak{K} if and only if \mathfrak{K} is σ -bounded and dominated by a family of at most \aleph_1 arrows.*

Proof. The “if” part is a special case of Theorem 3.5. Let \vec{u} be an ω_1 -Fraïssé sequence in \mathfrak{K} . Then \mathfrak{K} has the joint embedding property and the family $\{u_\alpha^\beta: \alpha \leq \beta < \omega_1\}$ is dominating in \mathfrak{K} . Fix $\vec{x} \in \sigma\mathfrak{K}$. Theorem 3.6 says that there exists an arrow of sequences $\vec{f}: \vec{x} \rightarrow \vec{u}$, so some u_α provides a bound for \vec{x} . Thus, every countable sequence is bounded in \mathfrak{K} . \square

3.4 The back-and-forth principle

Fix a category \mathfrak{K} and let \vec{u}, \vec{v} be Fraïssé sequences in \mathfrak{K} . We shall say that $\langle \vec{u}, \vec{v} \rangle$ satisfies the *back-and-forth principle* if for every α below the length of \vec{u} , for every arrow $f: u_\alpha \rightarrow \vec{v}$ there exists an isomorphism of sequences $h: \vec{u} \rightarrow \vec{v}$ such that $h \circ i_\alpha = f$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \vec{u} & \xrightarrow{h} & \vec{v} \\ i_\alpha \uparrow & \nearrow f & \\ u_\alpha & & \end{array}$$

Since there exists at least one arrow $f: u_0 \rightarrow \vec{v}$, this implies that $\vec{u} \approx \vec{v}$. It turns out that countable Fraïssé sequences always satisfy the back-and-forth principle. We shall see in Section 6.2 that this is not true for sequences of length ω_1 .

Theorem 3.9 (Uniqueness). *Assume that \vec{u}, \vec{v} are ω -Fraïssé sequences in a given category \mathfrak{K} . Assume further that $k, \ell < \omega$ and $f: u_k \rightarrow v_\ell$ is an arrow in \mathfrak{K} . Then there exists an isomorphism $F: \vec{u} \rightarrow \vec{v}$ in $\sigma\mathfrak{K}$ such that the diagram*

$$\begin{array}{ccc} \vec{u} & \xrightarrow{F} & \vec{v} \\ u_k^\infty \uparrow & & \uparrow v_\ell^\infty \\ u_k & \xrightarrow{f} & v_\ell \end{array}$$

commutes. In particular $\vec{u} \approx \vec{v}$.

Notice that in the above statement we do not assume that the given category has the amalgamation property.

Proof. We construct inductively arrows $f_n: u_{k_n} \rightarrow v_{\ell_n}$, $g_n: v_{\ell_n} \rightarrow u_{k_{n+1}}$, where $k_0 \leq \ell_0 < k_1 \leq \ell_1 < \dots$ and for each $n \in \omega$ the diagram

$$\begin{array}{ccccc} u_{k_n} & \xrightarrow{u_{k_n}^{k_{n+1}}} & u_{k_{n+1}} & \xrightarrow{u_{k_{n+1}}^{k_{n+2}}} & u_{k_{n+2}} \cdots \rightarrow \\ & \searrow f_n & \nearrow g_n & \searrow f_{n+1} & \nearrow g_{n+1} \\ & & v_{\ell_n} & \xrightarrow{v_{\ell_n}^{\ell_{n+1}}} & v_{\ell_{n+1}} \cdots \rightarrow \end{array}$$

commutes.

We start with $f_0 := f$, $k_0 := k$, $\ell_0 := \ell$, possibly replacing f by some arrow of the form $j_\ell^m \circ f$ to ensure that $k_0 \leq \ell_0$. Using property (A) of the sequence \vec{u} , find $k_1 > k_0$ and $g_1: v_0 \rightarrow u_{k_1}$ such that $g_1 \circ f = u_0^{k_1}$. Assume that f_m, g_m have already been constructed for $m \leq n$. Using the amalgamation of \vec{v} , find $\ell_{n+1} \geq k_{n+1}$ and an arrow $f_{n+1}: u_{k_{n+1}} \rightarrow v_{\ell_{n+1}}$ such that $f_{n+1} \circ g_n = v_{\ell_n}^{\ell_{n+1}}$. Now, using the amalgamation of \vec{u} , we

find $k_{n+2} > \ell_{n+1}$ and an arrow $g_{n+1}: v_{\ell_{n+1}} \rightarrow u_{k_{n+2}}$ such that $g_{n+1} \circ f_{n+1} = u_{k_{n+1}}^{k_{n+2}}$. By the induction hypothesis, $g_n \circ f_n = u_{k_n}^{k_{n+1}}$, therefore the above diagram commutes. This finishes the construction.

Finally, set $F = \{f_n\}_{n \in \omega}$ and $G = \{g_n\}_{n \in \omega}$. Then $F: \vec{u} \rightarrow \vec{v}$, $G: \vec{v} \rightarrow \vec{u}$ are morphisms of sequences and by a simple induction we show that

$$(*) \quad g_n \circ v_{\ell_m}^{\ell_n} \circ f_m = u_{k_m}^{k_{n+1}} \quad \text{and} \quad f_n \circ u_{k_{m+1}}^{k_n} \circ g_m = v_{\ell_m}^{\ell_n}$$

holds for every $m < n < \omega$. This shows that $F \circ G = \text{id}_{\vec{v}}$ and $G \circ F = \text{id}_{\vec{u}}$, therefore F is an isomorphism. The equality $v_0^\infty \circ f = F \circ u_0^\infty$ means that $v_0^{\ell_n} \circ f = f_n \circ u_0^{k_n}$ should hold for every $n \in \omega$. Fix $n > 0$. Applying $(*)$ twice (with $m = 0$ and $m = n - 1$ respectively), we get

$$f_n \circ u_0^{k_n} = f_n \circ g_{n-1} \circ v_0^{\ell_{n-1}} \circ f = v_{\ell_{n-1}}^{\ell_n} \circ v_0^{\ell_{n-1}} \circ f = v_0^{\ell_n} \circ f.$$

Thus $v_0^\infty \circ f = F \circ u_0^\infty$.

Finally, notice that, by property (U) of the sequence \vec{v} , for some $\ell < \omega$ there exists an arrow $f: u_0 \rightarrow v_\ell$, so applying the first part we see that $\vec{u} \approx \vec{v}$. \square

In general, there are examples of incomparable Fraïssé sequences of length ω_1 , so the back-and-forth principle may fail. However, in case of a continuous category the above arguments are easily generalized – this has already been done in [2]. Since our approach differs from that in [2], we shall give a detailed proof.

Theorem 3.10. *Let $\kappa > \aleph_0$ be a regular cardinal and let \mathfrak{K} be a κ -continuous category. Then every two Fraïssé sequences of length κ in \mathfrak{K} satisfy the back-and-forth principle.*

Proof. [...]

\square

TO DO:

- Straightforward applications: Fraïssé-Jónsson theory, reversed Fraïssé limits, the results of Droste & Göbel.
- Some examples (not too many!).

4 Fraïssé sequences and functors

Assume \mathfrak{K} is a category with amalgamation which has a Fraïssé sequence \vec{u} of an uncountable regular length κ , but the category itself is not κ -continuous. Then there is no direct way to show that \vec{u} is cofinal in $\text{Seq}_{\leq \kappa}(\mathfrak{K})$. The same applies to the back-and-forth principle. In fact, both cofinality and the back-and-forth principle for uncountable sequences sometimes fails. However, in some situations we can “move” our Fraïssé sequence to a different category showing its cofinality in the new category. This makes sense only if after moving the sequence we do not use too much information. In this section we discuss preservation of Fraïssé sequences with respect to functors and we introduce the notion of a Fraïssé sequence over a functor, which is useful in some applications. We explain our motivation below.

Let \mathfrak{L} be the category of nonempty compact metric lines with increasing quotients (such maps are automatically continuous). Consider two natural subcategories of \mathfrak{L} . Let $\mathfrak{K} \subseteq \mathfrak{L}$ have the same objects as \mathfrak{L} , while an arrow $f: X \rightarrow Y$ belongs to \mathfrak{K} if and only if it is right-invertible in the category of compact spaces. In other words, $f: X \rightarrow Y$ is an arrow in \mathfrak{K} iff f is an increasing quotient and there exists a continuous (necessarily increasing) map $j: Y \rightarrow X$ such that $f \circ j = \text{id}_Y$. Finally, let \mathfrak{K}_0 be the full subcategory of \mathfrak{K} whose objects are all 0-dimensional (metric compact) lines. The last category is dominated by a single arrow (see [...]) and hence it has a (reversed) Fraïssé sequence \vec{u} of length ω_1 . Now observe that \vec{u} is no longer Fraïssé in \mathfrak{K} , because it fails property (U). Further, \vec{u} has property (U) when considered in \mathfrak{L} , but it clearly fails (A) in \mathfrak{L} . On the other hand, \vec{u} satisfies the following variation of (A): given $\xi < \omega_1$ and an arrow $f: u_\xi \rightarrow y$ in \mathfrak{K} , there are $\eta \geq \xi$ and an arrow $g: y \rightarrow u_\eta$ in \mathfrak{L} so that $u_\xi^\eta = g \circ f$. Moreover, \mathfrak{K} satisfies the following version of amalgamation: given arrows $f: z \rightarrow x$, $g: z \rightarrow y$ such that $f \in \mathfrak{K}$ and $g \in \mathfrak{L}$, there are arrows $f' \in \mathfrak{L}$ and $g' \in \mathfrak{K}$ with $f' \circ f = g' \circ g$. Adding the fact that the Fraïssé sequence \vec{u} can be made continuous in \mathfrak{L} , it turns out that these properties are sufficient to conclude that \vec{u} is cofinal in \mathfrak{L} for all ω_1 -sequences from \mathfrak{K} . Since every isomorphism in \mathfrak{L} is also an isomorphism in \mathfrak{K}_0 , we shall further conclude that \vec{u} satisfies the back-and-forth principle in \mathfrak{K}_0 .

We shall come back to this example later.

TO DO:

- Preserving Fraïssé sequences.
- Functors with amalgamation.
- Back-and-forth Principle revisited.

5 Retractive pairs

In this section we describe a general construction on a given category, which is suitable for applications to the theory of Valdivia compacta and Banach spaces. This construction had been used by D. Scott [...] for getting certain models of unsigned λ -calculus.

We fix a category \mathfrak{K} . Define $\ddagger\mathfrak{K}$ to be the category whose objects are the objects of \mathfrak{K} and a morphism $f: X \rightarrow Y$ is a pair $\langle e, r \rangle$ of arrows in \mathfrak{K} such that $e: X \rightarrow Y$, $r: Y \rightarrow X$ and $r \circ e = \text{id}_X$. We set $e(f) := e$ and $r(f) := r$, so $f = \langle e(f), r(f) \rangle$. Given morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ in \mathfrak{K} , we define their composition in the obvious way:

$$g \circ f := \langle e(g), r(g) \rangle \circ \langle e(f), r(f) \rangle := \langle e(g) \circ e(f), r(f) \circ r(g) \rangle.$$

It is clear that this defines an associative operation on compatible arrows. Further, given an object $a \in \mathfrak{K}$, pair of the form $\langle \text{id}_a, \text{id}_a \rangle$ is the identity morphism in $\ddagger\mathfrak{K}$. Thus, $\ddagger\mathfrak{K}$ is indeed a category. Note that $f \mapsto e(f)$ defines a covariant functor e from $\ddagger\mathfrak{K}$ into \mathfrak{K} and $f \mapsto r(f)$ defines a contravariant functor $r: \ddagger\mathfrak{K} \rightarrow \mathfrak{K}$.

Now let $f: Z \rightarrow X$ and $g: Z \rightarrow Y$ be arrows in $\ddagger\mathfrak{K}$. We say that arrows $h: X \rightarrow W$, $k: Y \rightarrow W$ provide a *proper amalgamation* of f, g if $h \circ f = k \circ g$ and moreover $e(g) \circ r(f) = r(k) \circ e(h)$, $e(f) \circ r(g) = r(h) \circ e(k)$ hold. Translating it back to the original category \mathfrak{K} , this means that the following four diagrams commute:

$$\begin{array}{cccc} \begin{array}{ccc} W & \xleftarrow{e(k)} & Y \\ e(h) \uparrow & & \uparrow e(g) \\ X & \xleftarrow{e(f)} & Z \end{array} & \begin{array}{ccc} W & \xrightarrow{r(k)} & Y \\ r(h) \downarrow & & \downarrow r(g) \\ X & \xrightarrow{r(f)} & Z \end{array} & \begin{array}{ccc} W & \xrightarrow{r(k)} & Y \\ e(h) \uparrow & & \uparrow e(g) \\ X & \xrightarrow{r(f)} & Z \end{array} & \begin{array}{ccc} W & \xleftarrow{e(k)} & Y \\ r(h) \downarrow & & \downarrow r(g) \\ X & \xleftarrow{e(f)} & Z \end{array} \end{array}$$

We draw arrows $\xrightarrow{\quad}$ and $\xRightarrow{\quad}$ in order to indicate mono- and epimorphisms respectively. We shall say that $\ddagger\mathfrak{K}$ has *proper amalgamations* if every pair of arrows in $\ddagger\mathfrak{K}$ with common domain can be properly amalgamated in $\ddagger\mathfrak{K}$.

Below is a useful criterion for the existence of proper amalgamations.

Lemma 5.1. *Let \mathfrak{K} be a category and let f, g be arrows in $\ddagger\mathfrak{K}$ with the same domain. If $e(f), e(g)$ have a pushout in \mathfrak{K} then f, g can be properly amalgamated in $\ddagger\mathfrak{K}$.*

Proof. Let $h: X \rightarrow W$ and $k: Y \rightarrow W$ form a pushout of $e(f), e(g)$. Consider the following diagram:

$$\begin{array}{ccccc} & & Z & \xrightarrow{e(f)} & X & \xrightarrow{r(f)} & Z \\ & & \uparrow r(g) & & \nearrow \ell & & \uparrow r(g) \\ & & Y & \xrightarrow{k} & W & & Y \\ & & \uparrow e(g) & & \nearrow j & & \uparrow e(g) \\ Z & \xrightarrow{e(f)} & X & \xrightarrow{r(f)} & Z & & Z \end{array}$$

(Note: The diagram above is a simplified representation of the complex commutative diagram in the image, which includes identity maps id_X, id_Y and a central object W .)

The dotted arrows indicate unique morphisms completing appropriate diagrams, i.e. j is the unique arrow satisfying equations $j \circ h = e(g) \circ r(f)$, $j \circ k = \text{id}_Y$ and ℓ is the unique arrow satisfying equations $\ell \circ k = e(f) \circ r(g)$, $\ell \circ h = \text{id}_X$. Consequently, $\langle k, j \rangle$ and $\langle h, \ell \rangle$ are morphisms in $\ddagger \mathfrak{K}$. Set $s = r(f) \circ \ell$. Then

$$(1) \quad s \circ k = r(f) \circ \ell \circ k = r(f) \circ e(f) \circ r(g) = r(g) \quad \text{and} \quad s \circ h = r(f) \circ \ell \circ h = r(f).$$

Recall that $r(f)e(f) = \text{id}_Z = r(g)e(g)$. Since k, h is a pushout of $e(f), e(g)$, we deduce that s must be the unique arrow satisfying (1). Now let $t = r(g) \circ j$. Similar computations show that $t \circ k = r(g)$ and $t \circ h = r(f)$, therefore by uniqueness we deduce that $s = t$ or, in other words, $r(f) \circ \ell = r(g) \circ j$. This shows that the full diagram is commutative and hence $\langle k, j \rangle$ and $\langle h, \ell \rangle$ provide a proper amalgamation of f, g in the category $\ddagger \mathfrak{K}$. \square

As an example, if \mathfrak{K} is the category of nonempty sets, then Lemma 5.1 says that category $\ddagger \mathfrak{K}$ has proper amalgamations. We show below that a typical amalgamation in $\ddagger \mathfrak{K}$ may not be proper.

Example 5.2. Consider the category of nonempty sets \mathfrak{Set}^+ . Let a, b, c, d be pairwise distinct elements and set $Z = \{a\}$, $X = \{a, b\}$, $Y = \{a, c\}$ and $W = \{a, b, c\}$. We are going to define arrows $f: Z \rightarrow X$, $g: Z \rightarrow Y$, $h: X \rightarrow W$ and $k: Y \rightarrow W$ in the category $\ddagger \mathfrak{Set}^+$. Let $e(f), e(g), e(h)$ and $e(k)$ be the inclusion maps and let $r(f)$ and $r(g)$ be the obvious constant maps. Finally, let $r(h)(c) = a$ and $r(k)(b) = c$. This already defines $r(h)$ and $r(k)$, since these maps must be identity on the ranges of $e(h)$ and $e(k)$ respectively. It is clear that $h \circ f = k \circ g$, i.e. h, k amalgamate f, g in the category $\ddagger \mathfrak{Set}^+$. On the other hand, $e(g) \circ r(f)(b) = a$ and $r(k) \circ e(h)(b) = c$, therefore $e(g) \circ r(f) \neq r(k) \circ e(h)$. Note that actually $e(f) \circ r(g) = r(h) \circ e(k)$ holds, although redefining $r(h)(c)$ to b we can get $e(f) \circ r(g) \neq r(h) \circ e(k)$.

Let \mathfrak{K} be a fixed category. We have already seen that if left-invertible arrows have a pushout in \mathfrak{K} , then \mathfrak{K} has proper amalgamations. This is still insufficient to get cofinality for uncountable sequences. It turns out that usually $\ddagger \mathfrak{K}$ is not continuous, however one can consider the following weakening of continuity, which is good enough for applications.

We say that a sequence \vec{x} in $\ddagger \mathfrak{K}$ is *semicontinuous* if $e[\vec{x}]$ is continuous in \mathfrak{K} . The dual notion of semicontinuity with respect to the functor r can be obtained by considering $\ddagger \mathfrak{K}^{\text{op}}$ instead of $\ddagger \mathfrak{K}$.

Theorem 5.3. *Assume \mathfrak{K} is a category and \vec{u} is a semicontinuous Fraïssé sequence in $\ddagger \mathfrak{K}$ of a regular length κ . If $\ddagger \mathfrak{K}$ has proper amalgamations then for every semicontinuous sequence $\vec{x} \in \text{Seq}_{\leq \kappa}(\ddagger \mathfrak{K})$ there exists an arrow of sequences $\vec{f}: \vec{x} \rightarrow \vec{u}$.*

Proof. We construct a sequence of arrows $f_\alpha: x_\alpha \rightarrow u_{\varphi(\alpha)}$ so that $\{\varphi(\alpha)\}_{\alpha < \kappa}$ is strictly increasing and

$$(i) \quad \xi < \eta \implies u_{\varphi(\xi)}^{\varphi(\eta)} \circ f_\xi = f_\eta \circ x_\xi^\eta,$$

$$(ii) \quad \xi < \eta \implies r \left(u_{\varphi(\xi)}^{\varphi(\eta)} \right) \circ e(f_\eta) = e(f_\xi) \circ r \left(x_\xi^\eta \right).$$

We start with $f_0: x_0 \rightarrow u_{\varphi(0)}$ obtained from the fact that \vec{u} is Fraïssé. Fix an ordinal $\beta > 0$ and assume f_ξ have been defined for all $\xi < \beta$.

Suppose first that $\beta = \alpha + 1$. Find arrows $g: x_{\alpha+1} \rightarrow y$, $h: u_{\varphi(\alpha)} \rightarrow y$ which provide a proper amalgamation of $x_\alpha^{\alpha+1}$ and f_α . Using the amalgamation property of \vec{u} , find $\varphi(\alpha+1) > \varphi(\alpha)$ and an arrow $k: y \rightarrow u_{\varphi(\alpha+1)}$ such that $x_{\varphi(\alpha)}^{\varphi(\alpha+1)} = k \circ h$. The situation is described in the following diagram:

$$\begin{array}{ccccccc} \dots & \longrightarrow & u_{\varphi(\alpha)} & \xrightarrow{u_{\varphi(\alpha)}^{\varphi(\alpha+1)}} & u_{\varphi(\alpha+1)} & \longrightarrow & \dots \\ & & \uparrow f_\alpha & \searrow h & \uparrow k & & \\ & & & & y & & \\ & & & & \uparrow g & & \\ \dots & \longrightarrow & x_\alpha & \xrightarrow{x_\alpha^{\alpha+1}} & x_{\alpha+1} & \longrightarrow & \dots \end{array}$$

Define $f_{\alpha+1} = k \circ g$. By the induction hypothesis, it suffices to check (i) and (ii) with $\xi = \alpha$ and $\eta = \alpha + 1$. That (i) holds follows from the above commutative diagram. It remains to check that $r \left(u_{\varphi(\alpha)}^{\varphi(\alpha+1)} \right) \circ e(f_{\alpha+1}) = e(f_\alpha) \circ r \left(x_\alpha^{\alpha+1} \right)$. We know that

$$(1) \quad r(h) \circ e(g) = e(f_\alpha) \circ r(x_\alpha^{\alpha+1}),$$

because g, h properly amalgamate $x_\alpha^{\alpha+1}$ and f_α . On the other hand,

$$(2) \quad r \left(u_{\varphi(\alpha)}^{\varphi(\alpha+1)} \right) \circ e(k) = r(h) \circ r(k) \circ e(k) = r(h).$$

Using (1) and (2) we get

$$r \left(u_{\varphi(\alpha)}^{\varphi(\alpha+1)} \right) \circ e(f_{\alpha+1}) = r \left(u_{\varphi(\alpha)}^{\varphi(\alpha+1)} \right) \circ e(k) \circ e(g) = r(h) \circ e(g) = e(f_\alpha) \circ r(x_\alpha^{\alpha+1}).$$

Suppose now that β is a limit ordinal. Define $\varphi(\beta) = \sup_{\xi < \beta} \varphi(\xi)$. Then $\varphi(\beta) < \kappa$ is a limit ordinal, because the sequence $\{\varphi(\xi)\}_{\xi < \beta}$ was assumed to be strictly increasing. Thus, $u_{\varphi(\beta)}$ together with the cocone of arrows $\{\ell_\xi^\beta\}_{\xi < \beta}$ is the colimit of the sequence $\vec{v}: \beta \rightarrow \mathfrak{K}$, where $v_\xi = u_{\varphi(\xi)}$ and $v_\xi^\eta = e \left(x_{\varphi(\xi)}^{\varphi(\eta)} \right)$. Thus, by (i), there exists a unique arrow $e: x_\beta \rightarrow u_{\varphi(\beta)}$ such that the diagram

$$(3) \quad \begin{array}{ccc} u_{\varphi(\xi)} & \xrightarrow{e(v_\xi^\beta)} & u_{\varphi(\beta)} \\ \uparrow e(f_\xi) & & \uparrow e \\ x_\xi & \xrightarrow{e(x_\xi^\beta)} & x_\beta \end{array}$$

commutes in \mathfrak{K} for every $\xi < \beta$. Similarly, using (ii) and the fact that x_β is the colimit of $e[x \upharpoonright \beta]$ in \mathfrak{K} , we find a unique arrow $r: u_{\varphi(\beta)} \rightarrow x_\beta$ such that the diagram

$$(4) \quad \begin{array}{ccc} u_{\varphi(\xi)} & \xrightarrow{e(v_\xi^\beta)} & u_{\varphi(\beta)} \\ r(f_\xi) \downarrow & & \downarrow r \\ x_\xi & \xrightarrow{e(x_\xi^\beta)} & x_\beta \end{array}$$

commutes for each $\xi < \beta$. By uniqueness, we get $r \circ e = \text{id}_{x_\beta}$, so setting $f_\beta := \langle e, r \rangle$ we define an arrow in $\ddagger\mathfrak{K}$. Diagram (4) says that (ii) holds with $\eta := \beta$. Fix $\xi < \beta$. Diagram (3) says that $e(f_\beta) \circ e(x_\xi^\beta) = e(u_{\varphi(\xi)}^{\varphi(\beta)}) \circ e(f_\xi)$ holds. It remains to show that $r(x_\xi^\beta) \circ r(f_\beta) = r(f_\xi) \circ r(u_{\varphi(\xi)}^{\varphi(\beta)})$ holds for every $\xi < \beta$. For fix $\xi < \beta$ and let $p_\alpha := r(f_\xi) \circ r(u_{\varphi(\xi)}^\alpha)$ for $\alpha \geq \varphi(\xi)$. If $\alpha < \alpha'$ then

$$p_{\alpha'} \circ e(u_\alpha^{\alpha'}) = r(f_\xi) \circ r(u_\alpha^{\alpha'} \circ u_{\varphi(\xi)}^\alpha) \circ e(u_\alpha^{\alpha'}) = r(f_\xi) \circ r(u_{\varphi(\xi)}^\alpha) \circ r(u_\alpha^{\alpha'}) \circ e(u_\alpha^{\alpha'}) = p_\alpha.$$

Thus, by semicontinuity, there exists a unique arrow $q: u_{\varphi(\beta)} \rightarrow x_\beta$ in \mathfrak{K} satisfying $p_\alpha \circ e(u_\alpha^{\varphi(\beta)}) = q$ for $\alpha \in [\varphi(\xi), \varphi(\beta))$. On the other hand, $q := r(x_\xi^\beta) \circ r(f_\beta)$ satisfies this. In particular, setting $\alpha := \varphi(\xi)$, we get $r(x_\xi^\beta) \circ r(f_\beta) = r(f_\xi) \circ r(u_{\varphi(\xi)}^{\varphi(\beta)})$. This shows that the construction can be carried out until we reach the length of the sequence \vec{x} , which is assumed to be not greater than κ . This completes the proof. \square

In order to apply the above theorem, we shall need the following fact about sequences of left-invertible arrows.

Proposition 5.4. *Let \vec{x} be a continuous sequence in a category \mathfrak{K} and assume that each bonding arrow x_α^β is left-invertible in \mathfrak{K} . Then there exists a sequence \vec{y} in $\ddagger\mathfrak{K}$ such that*

$$\vec{x} = e[\vec{y}].$$

Proof. [...]

\square

We now turn to the question of homogeneity and uniqueness.

Theorem 5.5. *Let \mathfrak{K} be a category. Every two semicontinuous Fraïssé sequences of the same regular length in $\ddagger\mathfrak{K}$ satisfy the back-and-forth principle.*

Proof. Let \vec{u}, \vec{v} be semicontinuous Fraïssé sequences of length $\kappa = \text{cf } \kappa$ in $\ddagger\mathfrak{K}$ and let $f: u_0 \rightarrow v_0$ be given. We define inductively strictly increasing functions $\varphi: \kappa \rightarrow \kappa$, $\psi: \kappa \rightarrow \kappa$ and arrows $f_\alpha: u_{\psi(\alpha)} \rightarrow v_{\varphi(\alpha)}$, $g_\alpha: v_{\varphi(\alpha)} \rightarrow u_{\psi(\alpha+1)}$ in $\ddagger\mathfrak{K}$, so that

$$(i) \quad \alpha \leq \psi(\alpha) < \varphi(\alpha) \leq \psi(\alpha+1).$$

$$(ii) \quad g_\alpha \circ f_\alpha = u_{\psi(\alpha)}^{\psi(\alpha+1)} \text{ and } f_{\alpha+1} \circ g_\alpha = v_{\varphi(\alpha)}^{\varphi(\alpha+1)}.$$

$$(iii) \quad \xi < \eta \implies v_{\varphi(\xi)}^{\varphi(\eta)} \circ f_\xi = f_\eta \circ u_{\psi(\xi)}^{\psi(\eta)}.$$

We start with $\psi(0) = 0$, $\varphi(0) = \alpha$ and $f_0 = f$, where $0 < \alpha < \kappa$ is such that f is equivalent to $f_0: u_0 \rightarrow v_\alpha$.

Fix $\beta > 0$ and assume that $\varphi \upharpoonright \beta$, $\psi \upharpoonright \beta$, $\{f_\alpha\}_{\alpha < \beta}$ and $\{g_\alpha\}_{\alpha < \beta}$ have already been defined. In case where β is a successor ordinal, we proceed like in the proof of Theorem 3.10: using the fact that both sequences are Fraïssé, we first define f_β and later g_β , so that (i)–(iii) hold. So assume β is a limit ordinal.

Let $\varrho := \sup_{\alpha < \beta} \varphi(\alpha) = \sup_{\alpha < \beta} \psi(\alpha)$. Define $\varphi(\beta) := \psi(\beta) := \varrho$. Semicontinuity says that u_ϱ with the cocone of arrows $\{e(u_{\psi(\alpha)}^\varrho)\}_{\alpha < \beta}$ is the colimit of the sequence $e[u \circ (\psi \upharpoonright \beta)]$ in \mathfrak{K} . Similarly, v_ϱ is the colimit of $e[v \circ (\varphi \upharpoonright \beta)]$. Thus, there exist unique arrows $p: u_\varrho \rightarrow v_\varrho$, $q: v_\varrho \rightarrow u_\varrho$ in \mathfrak{K} such that the diagrams

$$(*) \quad \begin{array}{ccc} u_\varrho & \xrightarrow{p} & v_\varrho \\ \uparrow e(u_{\psi(\alpha)}^\varrho) & & \uparrow e(v_{\varphi(\alpha)}^\varrho) \\ u_{\psi(\alpha)} & \xrightarrow{e(f_\alpha)} & v_{\varphi(\alpha)} \end{array} \quad \begin{array}{ccc} u_\varrho & \xleftarrow{q} & v_\varrho \\ \uparrow e(u_{\psi(\alpha+1)}^\varrho) & & \uparrow e(v_{\varphi(\alpha)}^\varrho) \\ u_{\psi(\alpha+1)} & \xleftarrow{e(g_\alpha)} & v_{\varphi(\alpha)} \end{array}$$

commute for every $\alpha < \beta$. In particular, $q \circ p = \text{id}_{u_\varrho}$ and $p \circ q = \text{id}_{v_\varrho}$, so $f_\beta := \langle p, q \rangle$ is an arrow of \mathfrak{K} (in fact, it is an isomorphism between u_ϱ and v_ϱ). We need to check (iii) with $\eta := \beta$. For fix $\xi < \beta$. The first diagram in $(*)$ says that

$$(1) \quad e(v_{\varphi(\xi)}^\varrho) \circ e(f_\xi) = e(f_\beta) \circ e(u_{\psi(\xi)}^\varrho).$$

We need to show that

$$(2) \quad r(f_\xi) \circ r(v_{\varphi(\xi)}^\varrho) = r(u_{\psi(\xi)}^\varrho) \circ r(f_\beta).$$

We shall use the fact that v_ϱ is the colimit of $e[v \circ \varphi]$ restricted to the ordinal interval $[\xi, \beta)$. Given $\alpha \in [\xi, \beta)$, define

$$k_\alpha := r(f_\xi) \circ r(v_{\varphi(\xi)}^{\varphi(\alpha)}).$$

Then $k_\alpha: v_{\varphi(\alpha)} \rightarrow u_{\psi(\xi)}$ is an arrow in \mathfrak{K} . Observe that for $\alpha < \alpha'$ we have

$$\begin{aligned} k_{\alpha'} \circ e(v_{\varphi(\alpha)}^{\varphi(\alpha')}) &= r(f_\xi) \circ r(v_{\varphi(\alpha)}^{\varphi(\alpha')} \circ v_{\varphi(\xi)}^{\varphi(\alpha)}) \circ e(v_{\varphi(\alpha)}^{\varphi(\alpha')}) \\ &= r(f_\xi) \circ r(v_{\varphi(\xi)}^{\varphi(\alpha)}) \circ r(v_{\varphi(\alpha)}^{\varphi(\alpha')}) \circ e(v_{\varphi(\alpha)}^{\varphi(\alpha')}) \\ &= r(f_\xi) \circ r(v_{\varphi(\xi)}^{\varphi(\alpha)}) = k_\alpha. \end{aligned}$$

By the definition of a colimit, there exists a unique arrow $k: v_\varrho \rightarrow u_{\psi(\xi)}$ in \mathfrak{K} satisfying

$$(3) \quad k \circ e \left(v_{\varphi(\alpha)}^\varrho \right) = k_\alpha$$

for every $\alpha \in [\xi, \beta)$. Observe that, given $\alpha \in [\xi, \beta)$, we have

$$r(f_\xi) \circ r \left(v_{\varphi(\xi)}^\varrho \right) \circ e \left(v_{\varphi(\alpha)}^\varrho \right) = r(f_\xi) \circ r \left(v_{\varphi(\xi)}^{\varphi(\alpha)} \right) \circ r \left(v_{\varphi(\alpha)}^\varrho \right) \circ e \left(v_{\varphi(\alpha)}^\varrho \right) = k_\alpha.$$

By uniqueness, it follows that

$$k = r(f_\xi) \circ r \left(v_{\varphi(\xi)}^\varrho \right).$$

Now let

$$\ell := r \left(u_{\psi(\xi)}^\varrho \right) \circ q.$$

For each $\alpha \in [\xi, \beta)$, using the second diagram in (*), the first part of (ii) and (iii), we obtain

$$\begin{aligned} \ell \circ e \left(v_{\varphi(\alpha)}^\varrho \right) &= r \left(u_{\psi(\xi)}^\varrho \right) \circ e \left(u_{\psi(\alpha+1)}^\varrho \right) \circ e(g_\alpha) \\ &= r \left(u_{\psi(\xi)}^{\psi(\alpha+1)} \right) \circ r \left(u_{\psi(\alpha+1)}^\varrho \right) \circ e \left(u_{\psi(\alpha+1)}^\varrho \right) \circ e(g_\alpha) \\ &= r \left(u_{\psi(\xi)}^{\psi(\alpha+1)} \right) \circ e(g_\alpha) = r \left(u_{\psi(\xi)}^{\psi(\alpha)} \right) \circ r \left(u_{\psi(\alpha)}^{\psi(\alpha+1)} \right) \circ e(g_\alpha) \\ &= r \left(u_{\psi(\xi)}^{\psi(\alpha)} \right) \circ r(f_\alpha) \circ r(g_\alpha) \circ e(g_\alpha) \\ &= r \left(f_\alpha \circ u_{\psi(\xi)}^{\psi(\alpha)} \right) = r \left(v_{\varphi(\xi)}^{\varphi(\alpha)} \circ f_\xi \right) \\ &= r(f_\xi) \circ r \left(v_{\varphi(\xi)}^{\varphi(\alpha)} \right) = k_\alpha. \end{aligned}$$

It follows that ℓ satisfies (3), therefore $k = \ell$, by uniqueness. This means that (2) is true. Recalling that $\varrho = \varphi(\beta) = \psi(\beta)$, we have proved that $f_\beta \circ u_{\psi(\xi)}^{\psi(\beta)} = v_{\varphi(\xi)}^{\varphi(\beta)} \circ f_\xi$, i.e. condition (iii) is satisfied with $\eta := \beta$. We still need to define $\psi(\beta+1)$ and g_β . Since f_β is invertible in $\ddagger\mathfrak{K}$ (its inverse is $\langle q, p \rangle$), we may set $\psi(\beta+1) := \varrho + 1$ and $g_\beta := u_\varrho^{\varrho+1} \circ f_\beta^{-1}$. Clearly, condition (i) and the first part of (ii) are fulfilled. The second part of (ii) with α replaced by β is taken care of in the successor step $\beta + 1$, which we have already justified.

Finally, like in the proof of Theorem 3.10, we deduce from conditions (i)–(iii) that $\vec{f} = \{f_\alpha\}_{\alpha < \kappa}$ and $\vec{g} = \{g_\alpha\}_{\alpha < \kappa}$ are arrows of sequences in $\ddagger\mathfrak{K}$ such that \vec{f} extends f and, by condition (ii), these arrows are isomorphisms in the category of sequences. \square

Corollary 5.6. *Let κ be a regular cardinal and let \mathfrak{K} be a κ -continuous category. Assume that $\ddagger\mathfrak{K}$ has a Fraïssé sequence of length κ . Then $\ddagger\mathfrak{K}$ also has a semicontinuous Fraïssé sequence of length κ .*

Proof. [...]

□

TO DO: Fill the gaps and add more comments.

6 Applications

In this section we collect few applications of our results – mainly of those from Section 5 – to Banach spaces, Valdivia compacta and linearly ordered sets. We also describe a natural category of binary trees which has many pairwise incomparable Fraïssé sequences of length ω_1 .

6.1 Universal Banach spaces

[History ...]

We first discuss a Banach space which is universal for linear isometric embeddings. Its existence, under the continuum hypothesis, actually follows from the results of Droste & Göbel [2].

Let $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ denote the category whose objects are separable Banach spaces and arrows are linear isometries.

Lemma 6.1. $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ has the amalgamation property.

Proof. Fix $X, Y, Z \in \mathfrak{B}_{\aleph_0}^{\text{iso}}$ and fix linear isometric embeddings $f: Z \rightarrow X$ and $g: Z \rightarrow Y$. Without loss of generality, we may assume that f and g are inclusions, i.e. $Z \subseteq X$ and $Z \subseteq Y$. We may also assume that $X \cap Y = Z$. Now let W be the formal algebraic sum of X and Y , i.e. $W = \{x + y: x \in X, y \in Y\}$ and $x + y = x' + y'$ whenever $x - x' = y' - y$. Let G be the convex hull of B_X and B_Y , where B_X, B_Y denote the closed unit balls of X and Y respectively. Let $\|\cdot\|$ be the norm induced by the Minkowski functional of G . Then the completion of $\langle W, \|\cdot\| \rangle$ is a separable Banach space which provides an amalgamation of f and g in $\mathfrak{B}_{\aleph_0}^{\text{iso}}$. □

Clearly, $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ has an initial object, the zero space. Thus, the joint embedding property follows from amalgamation. The next statement is rather clear.

Lemma 6.2. $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ is σ -continuous.

Theorem 6.3. Assume CH. There exists a Banach space V of density \aleph_1 such that every Banach space of density $\leq \aleph_1$ is linearly isometric to a subspace of V and every linear isometry $T: X \rightarrow Y$ between separable subspaces of V can be extended to a linear isometry of V . Moreover, the space V is unique, up to a linear isometry.

Proof. Assuming the continuum hypothesis, there are only \aleph_1 many isometric types of separable Banach spaces and there are only \aleph_1 many types of linear isometries. Thus, by Lemmas 6.1 and 6.2 and by Theorem 3.5, $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ has a Fraïssé sequence \vec{u} of length ω_1 . We may further assume that this sequence is continuous. Let V be the colimit of \vec{u} in the category of all Banach spaces.

Fix a Banach space X of density $\leq \aleph_1$. We can write $X = \bigcup_{\alpha < \omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha < \omega_1}$ is an increasing chain of closed separable subspaces of X such that $X_\delta = \text{cl}(\bigcup_{\xi < \delta} X_\xi)$ for every limit ordinal $\xi < \omega_1$. Translating it to the language of category theory, we obtain a continuous ω_1 -sequence in $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ whose colimit, in the category of all Banach spaces, is X . By Theorem 3.7, there is an arrow of sequences $F: \vec{x} \rightarrow \vec{u}$. This arrow has a colimit in the category of all Banach spaces, which is just a linear isometric embedding of X into V .

The second statement is obtained by the back-and-forth principle, using the continuity of \vec{u} . \square

We do not know much about the space V from the above theorem, although we remark below that it cannot be isometric to any $\mathcal{C}(K)$ space.

Proposition 6.4. *Let K be a compact space which contains at least two points. Then there exists a linear isometry $T: X \rightarrow Y$ between 1-dimensional subspaces of $\mathcal{C}(K)$, which cannot be extended to a linear isometry of $\mathcal{C}(K)$.*

Proof. Fix $a \neq b$ in K . Let X consist of all constant functions on K . Let $R: \mathcal{C}(\{a, b\}) \rightarrow \mathcal{C}(K)$ be a regular extension operator for the inclusion $\{a, b\} \subseteq K$. That is, R is a linear operator which assigns to each $f \in \mathcal{C}(\{a, b\})$ its extension $Rf \in \mathcal{C}(K)$ so that $R1 = 1$ and $Rf \geq 0$ whenever $f \geq 0$. For example, let $(Rf)(t) = \varphi(t)f(a) + (1 - \varphi(t))f(b)$, where $\varphi: K \rightarrow [0, 1]$ is a continuous function such that $\varphi(a) = 1$ and $\varphi(b) = 0$ (which exists by Urysohn's Lemma). Note that R is an isometric embedding of $\mathcal{C}(\{a, b\})$ into $\mathcal{C}(K)$.

Now define $T: X \rightarrow \mathcal{C}(K)$ by $T1 = R1_{\{a\}}$, where $1_{\{a\}}$ is the function which takes value 1 at a and value 0 at b . Let $Y = T[X] = \{\lambda R1_{\{a\}}: \lambda \in \mathbb{R}\}$. Then T is an isometry.

Suppose $\bar{T}: \mathcal{C}(K) \rightarrow \mathcal{C}(K)$ is a linear isometry extending T . Let $v = (\bar{T})^{-1}[R1_{\{b\}}]$. Then $\|v\| = 1$. By compactness, there exists $t \in K$ such that $|v(t)| = 1$. Let $\alpha = v(t)$ and consider $u = R(\alpha 1_{\{a\}} + 1_{\{b\}}) = \alpha R1_{\{a\}} + R1_{\{b\}}$. Notice that $\|u\| = 1$, while $\|(\bar{T})^{-1}(u)\| = \|\alpha 1_K + v\| \geq |\alpha + v(t)| = 2$. Hence $\|(\bar{T})^{-1}\| \geq 2$, a contradiction. \square

It is well known that, under the continuum hypothesis, the compact space $\omega^* := \beta\omega \setminus \omega$ is the universal continuous preimage for compact spaces of weight $\leq \aleph_1$. Moreover, ω^* is homogeneous with respect to quotients on metrizable compacta. In fact, these statements hold without any extra set-theoretic assumptions, see [...]. Now let $W = \ell_\infty/c_0 = \mathcal{C}(\omega^*)$. Then W contains a linear isometric copy of any Banach space of density $\leq \aleph_1$. Indeed, if X is such a space then $X \subseteq \mathcal{C}(K)$, where K is the unit ball of the

dual space X^* . Now, a quotient map $f: \omega^* \rightarrow K$ induces a linear isometric embedding of $\mathcal{C}(K)$ into $\mathcal{C}(\omega^*) = \ell_\infty/c_0$. Let V be the space from Theorem 6.3. By the above proposition, V is not linearly isometric to any space of the form $\mathcal{C}(K)$, where K is a compact space. In particular, $V \neq \ell_\infty/c_0$. We do not know whether V could be just *isomorphic* to $\mathcal{C}(K)$ for some compact space K of weight $\aleph_1 = 2^{\aleph_0}$. Motivated by topological properties of the compact space $\beta\omega \setminus \omega$, we ask:

Question 6.5. Does there exist in ZFC a Banach space V of density 2^{\aleph_0} which has properties like the space in Theorem 6.3 ?

We now turn to a more special class of Banach spaces, namely Banach spaces with projectional resolutions. From this point on, we consider the category \mathfrak{B}_{\aleph_0} whose objects are again all separable Banach spaces and arrows are linear operators of norm ≤ 1 . We shall apply the results of Section 5. Our aim is to obtain a universal Banach space with a projectional resolution of the identity or, euquivalently, with a countably norming Markushevich basis.

Lemma 6.6. *Let $f: Z \rightarrow X$, $g: Z \rightarrow Y$ be left-invertible arrows in \mathfrak{B}_{\aleph_0} . Then f, g have a pushout in \mathfrak{B}_{\aleph_0} .*

Proof. We may assume that $X = Z \oplus X_1$, $Y = Z \oplus Y_1$ and f, g are the canonical embeddings. Let $W = Z \oplus X_1 \oplus Y_1$ and let G be the convex hull of $B_X \cup B_Y$, where B_X, B_Y denote the closed unit balls of X and Y respectively. Let $\|\cdot\|$ be the Minkowski functional of G . Then W becomes a Banach space and there are natural embeddings $f': X \rightarrow W$, $g': Y \rightarrow W$. It is easy to see that f', g' is a pushout of f, g . Indeed, let $p: X \rightarrow U$, $q: Y \rightarrow U$ be arrows of \mathfrak{B}_{\aleph_0} such that $p \circ f = q \circ g$. There is a unique linear transformation $h: W \rightarrow U$ such that $h \circ f' = p$ and $h \circ g' = q$. Observe that $p[B_X] \subseteq B_U$ and $q[B_Y] \subseteq B_U$, therefore $h[G] \subseteq B_U$. It follows that $\|h\| \leq 1$. \square

Lemma 6.7. *Let $\vec{a}: \omega \rightarrow \mathfrak{B}_{\aleph_0}$ be such that a_m^n is left-invertible in \mathfrak{B}_{\aleph_0} for every $m < n < \omega$. Then \vec{a} has the colimit in \mathfrak{B}_{\aleph_0} .*

Proof. The assumption that a_m^n is left-invertible is superfluous: it suffices to assume that each a_m^n is a linear isometric embedding. We may assume that $a_m \subseteq a_n$ for $m < n$ and that a_m^n is just the inclusion for $m < n$. Let $X_0 = \bigcup_{n < \omega} a_n$. Then X_0 is a normed linear space, endowed with the obvious norm. Let X be the completion of X_0 . We claim that X together with the cocone of inclusions $a_n^\infty: a_n \rightarrow X$ is the colimit of \vec{a} . For fix an object $y \in \mathfrak{B}_{\aleph_0}$ and a family of arrows $\{f_n\}_{n < \omega}$ such that $f_n: a_n \rightarrow y$ and $f_n = f_m \circ a_n^m$ for every $n < m < \omega$. There is a unique function $h_0: X_0 \rightarrow y$ satisfying $h_0 \circ a_n^\infty = f_n$. It is clear that h_0 is linear and $\|h_0\| \leq 1$. Thus, h_0 has a unique extension to a continuous linear transformation $h: X \rightarrow y$. Clearly, $\|h\| \leq 1$ and $h \circ a_n^\infty = f_n$ holds for every $n < \omega$. \square

We now have all ingredients needed to construct a universal Banach space with a PRI.

Theorem 6.8. *Assume CH. There exists a Banach space E with a PRI and of density \aleph_1 , which has the following properties:*

- (a) *The family $\{X \subseteq E: X \text{ is 1-complemented in } E\}$ is, modulo linear isometries, the class of all Banach spaces of density $\leq \aleph_1$ with a PRI.*
- (b) *Given separable subspaces $X, Y \subseteq E$, norm one projections $P: E \rightarrow X$, $Q: E \rightarrow Y$ and a linear isometry $T: X \rightarrow Y$, there exist a linear isometry $S: E \rightarrow E$ extending T and satisfying $P \circ S^{-1} = T^{-1} \circ Q$.*

Moreover, the above properties describe the space E uniquely, up to a linear isometry.

Proof. [...]

□

6.2 Binary trees

In this subsection we describe the announced example of a category of trees which has many pairwise non-equivalent ω_1 -Fraïssé sequences.

By a *tree* we mean a partially ordered set $\langle T, \leq \rangle$ which is a meet semilattice, i.e. every two elements of T have the greatest lower bound, and for every $t \in T$ the interval $\{x \in T: x < t\}$ is well ordered. Every tree T has a single minimal element 0_T , called the *root* of T . An *immediate successor* of $t \in T$ is an element $s > t$ such that no $x \in T$ satisfies $t < x < s$. A *subtree* of a tree T is a subset $S \subseteq T$ which is closed under the meet operation. A tree T is *binary* if every $t \in T$ has at most two immediate successors. We shall denote by $\max T$ the set of all maximal elements of T . A tree T is *bounded* if for every $x \in T$ there is $t \in \max T$ such that $x \leq t$. Recall that an *initial segment* of a poset $\langle T, \leq \rangle$ is a subset A of T satisfying $\{x \in T: x \leq t\} \subseteq A$ for every $t \in A$. A subset A of T is *closed* if $\sup C \in A$ for every chain $C \subseteq A$. This is equivalent to saying that A is closed with respect to the *interval topology* on T generated by intervals of the form $[0, t]$ and $(s, t]$, where $s < t$.

We define the category \mathfrak{T}_2 as follows. The objects of \mathfrak{T}_2 are nonempty countable bounded binary trees. An arrow from $T \in \mathfrak{T}_2$ into $S \in \mathfrak{T}_2$ is a semilattice embedding $f: T \rightarrow S$ such that $f[T]$ is a closed initial segment of S .

A tree T is *healthy* if every element of $T \setminus \max(T)$ has at least two immediate successors and for every $t \in T$ and $\alpha < \text{ht}(T)$ there exists $s \geq t$ such that $\text{Lev}_T(s) \geq \alpha$. An example of a healthy tree of height ω_1 is

$$T = \{x \in 2^{<\omega_1}: |\{\alpha: x(\alpha) = 1\}| < \aleph_0\}.$$

Note that all levels of T are countable. Setting $T_\alpha = \{x \in T: \text{dom}(x) \subseteq \alpha + 1\}$, we obtain an inductive sequence $\{T_\alpha\}_{\alpha < \omega_1}$ in the category \mathfrak{T}_2 . More generally, if S is any binary tree of height ω_1 whose all levels are countable then, setting

$$S_\alpha = \{x \in S: \text{the order type of } [0, x) \text{ is } \leq \alpha\}$$

we obtain an inductive sequence \vec{S} in \mathfrak{T}_2 , where each S_α^β is the inclusion, which is an arrow in \mathfrak{T}_2 . We shall say that \vec{S} is the *natural decomposition* of S .

Lemma 6.9. *Let $V \in \mathfrak{T}_2$ be a healthy tree of height $\alpha + 1$. Then every $T \in \mathfrak{T}_2$ with $\text{ht}(T) \leq \alpha + 1$ is isomorphic to a closed initial segment of V .*

Proof. Denote by \mathfrak{M} the class of all nonempty bounded binary trees T of height $\leq \alpha + 1$ such that $\max(T)$ is finite. Given such a tree T , write $\max(T) = \{w_0, \dots, w_{m-1}\}$ and define inductively $T_0 := [0, w_0]$ and $T_k := [0, w_k] \setminus (T_0 \cup \dots \cup T_{k-1})$. Then $\mathcal{D} = \{T_0, \dots, T_{m-1}\}$ is a natural decomposition of T into connected chains, induced by the enumeration of $\max(T)$.

Let \mathfrak{L} be a category whose objects are pairs $\langle T, \mathcal{D} \rangle$, where $T \in \mathfrak{M}$ and \mathcal{D} is a natural decomposition into connected chains induced by an enumeration of $\max(T)$, as described above. Given $\langle T, \mathcal{D} \rangle, \langle S, \mathcal{F} \rangle \in \mathfrak{L}$, an arrow in \mathfrak{L} is a tree embedding $f: T \rightarrow S$ with the following properties:

- (a) For each $D \in \mathcal{D}$ there is $F \in \mathcal{F}$ such that $f[D]$ is an initial segment of F .
- (b) For each $F \in \mathcal{F}$ there exists at most one D such that $f[D] \subseteq F$.

It is clear that these properties are preserved under the usual composition, so \mathfrak{L} is indeed a category. Now consider the given healthy tree V with $\max(V) = \{e_n\}_{n \in \omega}$ and define $V^n = \bigcup_{i \leq n} [0, e_i]$. Let \mathcal{D}^n be the decomposition of V^n induced by the enumeration $\{e_0, \dots, e_n\}$ of $\max(V^n)$. Then $\langle V^n, \mathcal{D}^n \rangle \in \mathfrak{L}$ and $\vec{V} = \{\langle V^n, \mathcal{D}^n \rangle\}_{n \in \omega}$ is an inductive sequence in \mathfrak{L} . We claim that:

- (1) \vec{V} is a Fraïssé sequence in \mathfrak{L} which has property (E).
- (2) If \vec{T} is an inductive sequence in \mathfrak{L} and $\vec{f} = \{f_n\}_{n \in \omega}$ is an embedding of \vec{T} into \vec{V} then the embedding $f: T \rightarrow V$ induced by \vec{f} has the property that $f[T]$ is a closed initial segment of V .

We first show (2): Fix $y \in V \setminus f[T]$. Find m and $D \in \mathcal{D}_m$ such that $y \in D$. Let \mathcal{S}_n be the natural decomposition of T_n , $\mathcal{S} = \bigcup_{n \in \omega} \mathcal{S}_n$. Then there is at most one $S \in \mathcal{S}$ such that $f[S] \subseteq D$. Moreover $f[S]$ is closed in D , so $D \setminus f[S]$ is a neighborhood of y disjoint from $f[T]$.

For the proof of (1), fix $\langle T, \mathcal{F} \rangle \in \mathfrak{L}$ and assume $f: T \rightarrow V_n$ is an arrow in \mathfrak{L} . Let $T \subseteq T'$ and let $\mathcal{F}' \supseteq \mathcal{F}$ so that the inclusion $T \subseteq T'$ is an arrow between $\langle T, \mathcal{F} \rangle$ and $\langle T', \mathcal{F}' \rangle$. Without loss of generality, we may assume that $\mathcal{F}' = \mathcal{F} \cup \{A\}$, i.e. T' differs from T by only one new branch. Let $a = \min A$. Then, by the definition of the natural decomposition, a has an immediate predecessor $c \in T$ (note that $0 \in T$ so $a > 0$). Find $F \in \mathcal{F}$ such that $c \in F$. Then $f(c)$ has exactly two immediate successors in V and at most one belongs to $f[T]$, since c has only two immediate successors in T . Let

$d \in V \setminus f[T]$ be an immediate of $f(c)$. Find a big enough $m > n$ so that there exists $D \in \mathcal{D}_m$ with $d \in D$. Since each maximal element of V has height α , D is a cofinal branch in V and hence A can be (uniquely) embedded into D as an initial segment. This embedding defines an extension $\vec{f}: T' \rightarrow V_m$ of f . Since \mathfrak{L} has a minimal object, this shows that \vec{V} is Fraïssé and satisfies (E).

Finally, fix $T \in \mathfrak{T}_2$ with $\text{ht}(T) \leq \alpha + 1$. Decompose T into an inductive ω -sequence, according to a fixed enumeration of $\max(T)$. Claims (1) and (2) say that T can be embedded into V as a closed initial subtree. This completes the proof. \square

Theorem 6.10. *Assume U is a healthy binary tree of height ω_1 , whose all levels are countable. Let \vec{U} be the natural \mathfrak{T}_2 -decomposition of U . Then \vec{U} is a Fraïssé sequence in \mathfrak{T}_2 which has the extension property. In particular, \mathfrak{T}_2 has both the amalgamation and the joint embedding property.*

Proof. Note that \mathfrak{T}_2 has a minimal object, namely the one-element tree. Clearly, such a tree embeds into U_0 . It suffices to show that \vec{U} satisfies (E) – it will then follow that \vec{U} is a Fraïssé sequence. Since there exists a healthy binary tree of height ω_1 with countable levels, we shall be able to conclude that \mathfrak{T}_2 has a Fraïssé sequence satisfying (E) and consequently \mathfrak{T}_2 has both the amalgamation and the joint embedding property. Thus, it remains to show that \vec{U} has the extension property.

Fix $\alpha < \omega_1$ and fix an arrow $f: T \rightarrow U_\alpha$ in \mathfrak{T}_2 and assume that T is a closed initial subtree of S , i.e. the inclusion $T \subseteq S$ is an arrow of \mathfrak{T}_2 . Fix $\alpha < \omega_1$ so that $\text{ht}(S) < \alpha$. Let $\{s_n: n \in \omega\}$ enumerate all minimal elements of $S \setminus T$. Let t_n be the immediate predecessor of s_n . Then $t_n \in T$. Recall that $f(t_n)$ has exactly two immediate successors in U_α and at least one of them does not belong to $f[T]$, since otherwise t_n would already have two immediate successors in T . Let y_n be an immediate successor of $f(t_n)$ which does not belong to $f[T]$. Let $V_n = \{y \in U_\alpha: y \geq y_n\}$. Then V_n is a healthy binary tree of countable height. By Lemma 6.9 we can embed $G_n = \{s \in S: s \geq s_n\}$ onto a closed initial segment of V_n . Combining all these embeddings, we obtain an extension $\vec{f}: S \rightarrow U_\alpha$ of f . We claim that $\vec{f}[S]$ is closed in U_α . Indeed, if $C \subseteq S$ is a chain and $C \not\subseteq T$ then $c \geq s_n$ for some $c \in C$ and for some n . Thus $\sup C$ exists in S and hence $\sup \vec{f}[C]$ exists in $\vec{f}[G_n] \subseteq \vec{f}[S]$. \square

Theorem 6.11. *Assume \vec{U} and \vec{V} are ω_1 -Fraïssé sequences in \mathfrak{T}_2 inducing healthy trees U and V respectively. Assume further that $F: \vec{U} \rightarrow \vec{V}$ is an arrow of sequences. Then the trees U, V are isomorphic.*

Proof. Let $f: U \rightarrow V$ be the embedding induced by F , i.e. assuming both \vec{U}, \vec{V} are chains of trees, f is the union of arrows $f_\alpha: U_\alpha \rightarrow V_{\varphi(\alpha)}$ in \mathfrak{T}_2 , where $\varphi: \omega_1 \rightarrow \omega_1$ is an increasing function. We claim that $f[U]$ is closed in $V_{\varphi(\alpha)}$. Suppose otherwise and fix a sequence $x_0 < x_1 < \dots$ in U such that $y = \sup_{n \in \omega} f(x_n) \notin f[U]$. Find $\beta < \omega_1$ such that $\{x_n: n \in \omega\} \subseteq U_\beta$ and $y \in V_{\varphi(\beta)}$. Then $f(x_n) = f_\beta(x_n)$ and $y \notin f_\beta[U_\beta]$, which shows that f_β is not an arrow in \mathfrak{T}_2 , a contradiction.

We finally claim that $V = f[U]$, which of course shows that f is an isomorphism. Suppose $V \neq f[U]$ and fix a minimal element $y \in V \setminus f[U]$. Find $\alpha < \omega_1$ such that $y \in V_{\varphi(\alpha)}$. Since $f[U]$ is closed in $V_{\varphi(\alpha)}$, y has an immediate predecessor, say $v = f(u)$. Let a, b be the two immediate successors of u in U (which exist, because U is healthy). Then either $y = f(a)$ or $y = f(b)$, because V is binary and $f[U]$ is an initial segment of U . This is a contradiction. \square

It can be easily shown that every tree induced by an ω_1 -Fraïssé sequence in \mathfrak{T}_2 is healthy, therefore this assumption can be removed from the above statement.

Recall that sequences \vec{a} and \vec{b} of the same length are *comparable* if there exists an arrow of sequences \vec{f} such that either $\vec{f}: \vec{a} \rightarrow \vec{b}$ or $\vec{f}: \vec{b} \rightarrow \vec{a}$. Otherwise, we say that \vec{a} and \vec{b} are *incomparable*.

Corollary 6.12. *There exist two incomparable ω_1 -Fraïssé sequences in \mathfrak{T}_2 .*

Proof. Let $U = \{x \in 2^{<\omega_1} : |x^{-1}(1)| < \aleph_0\}$ and let V be a healthy binary Aronszajn tree. Clearly, U and V are not isomorphic. Both trees can be naturally decomposed into ω_1 -sequences \vec{U} and \vec{V} respectively. By Theorem 6.10, \vec{U} and \vec{V} are Fraïssé sequences. By Theorem 6.11, these sequences are incomparable. \square

TO DO:

- A universal Valdivia compact of weight \aleph_1 , under CH.
- A universal linearly ordered Valdivia compact; retractive linearly ordered sets.

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